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EIGENVALUE PROBLEMS FOR STURM–LIOUVILLE EQUATIONS WITH TRANSMISSION CONDITIONS

BY

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ABSTRACT

In this study, we consider a Sturm–Liouville type boundary-value problem with eigenparameter-dependent boundary conditions and with two supplementary transmission conditions at one inner point of a finite interval under consideration. We modify some techniques of classical Sturm–Liouville theory and suggest a new approach for the investigation of eigenvalues and eigenfunctions of this type of boundary-value problem.

1. Introduction

It is well-known that many topics in mathematical physics lead to Sturm–Liouville type boundary-value problems. With historical roots in the application of Fourier series to heat flow, Sturm–Liouville theory is one of the most extensively developing fields in theoretical and applied mathematics. It should be mentioned that, in recent years, many important results in this field have been obtained for such boundary-value type problems, where eigenparameters appear not only in the differential equations, but also in the boundary conditions. The literature on such problems is voluminous and we refer to [1, 2, 3, 14] and references therein. In particular, [2, 3, 4, 8, 9, 11] contains many references to problems in physics and mechanics.

The main aim of this paper is to extend and generalize some approaches and results of this kind of Sturm–Liouville problem to similar types of problems,

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but with transmission conditions. Namely, we investigate the Sturm–Liouville equation

$$\tau u := -u'' + q(x)u = \lambda u, \tag{1.1}$$

to hold in a finite interval (-1,1) except at one inner point $x = 0$, subject to the eigenparameter-dependent boundary conditions

$$u(-1) = 0, \tag{1.2}$$

$$(\lambda - a)u'(1) + \lambda b u(1) = 0, \tag{1.3}$$

and transmission conditions at the inner point $x = 0$

$$u(-0) = h_1 u(+0), \tag{1.4}$$

$$u'(-0) = h_2 u'(+0), \tag{1.5}$$

where λ is a complex eigenvalue parameter; the function $q(x)$ is real-valued and continuous in each of the intervals $[-1, 0)$ and $(0, 1]$, and has finite limits $q(\pm 0) := \lim_{x\to\pm 0} q(x)$; a, b, h_1, h_2 are real coefficients. Throughout this study we assume that $h_1h_2 > 0$ and $ab > 0$. Consequently, we are interested in three types of generalizations of classical Sturm–Liouville problems:

First, we wish to consider more general differential equations, for which the potential $q(x)$ may have a discontinuity at one inner point of the considered interval.

A second generalization is to allow boundary conditions depending on an eigenparameter.

The final generalization that we discuss in this study concerns the point of discontinuity $x = 0$, at which two supplementary transmission conditions are given.

Note that some special cases of the problem (1.1)–(1.5) arise after an application of the method of separation of variables to the varied assortment of physical problems. For example, eigenparameter-dependent boundary conditions in the form (1.2)–(1.3) arise upon separation of variables in some heat transfer problems (see, for example, [3, 4, 8, 9]). Also, some boundary-value problems with transmission conditions arise in heat and mass transfer problems (see, for example, [4]), in vibrating string problems when the string is loaded additionally with point masses (see, for example, [9]), in diffraction problems (see, for example, [13]), etc. Properties such as isomorphism, coerciveness with respect to the spectral parameter, completeness and Abel bases of a system of root functions, distributions of eigenvalues of some discontinuous boundary value problems with transmission conditions and applications to the corresponding initial-boundary-value problems for parabolic equations have been investigated in [5, 6, 7, 14]. Also, some problems with transmission conditions which arise in mechanics (thermal conduction problem for a thin laminated plate) were studied in the article [11].

For a suitable operator-theoretic formulation of the considered problem (1.1)–(1.5), we introduce a new equivalent inner product on $H := L_2(-1,1) \oplus C$ by

$$\langle F, G\rangle = \int_{-1}^{0} F_1(x)\overline{G_1}(x)dx + h_1h_2\int_0^1 F_1(x)\overline{G_1}(x)dx + \frac{h_1h_2}{ab}F_2\overline{G_2} \tag{1.6}$$

for $F = (F_1(x), F_2), G = (G_1(x), G_2) \in H$.

In this Hilbert space H we define a linear operator A: $D(A) \to H$ by $A(F_1(x), F_2) = (\tau F_1, aF_1'(1))$, where the domain $D(A)$ is defined as the set of all $F = (F_1(x), F_2) \in H$ which satisfy the following conditions:

(i) $F_1(x)$ and $F_1'(x)$ are absolutely continuous on both intervals $[-1, 0)$ and $(0, 1]$, and have finite limits

$$F_1(\pm 0) = \lim_{x\to\pm 0} F_1(x) \quad \text{and} \quad F_1'(\pm 0) = \lim_{x\to\pm 0} F_1'(\pm 0),$$

respectively;

(ii) $\tau F_1 \in L_2(-1,1)$;

(iii) $F_1(-1) = 0, F_1(-0) = h_1F_1(+0), F_1'(-0) = h_2F_1'(+0)$;

(iv) $F_2 = bF_1(1) + F_1'(1)$.

So we can pose the considered problem (1.1)–(1.5) in the operator-equation form as

$$AU = \lambda U,$$

where $U = (u(x), bu(1) + u'(1)) \in D(A)$.

Naturally, by eigenvalues and eigenfunctions of the problem (1.1)–(1.5) we mean eigenvalues and first components of corresponding eigenelements of the operator A, respectively.

In our previous paper [12], by applying this operator-theoretic formulation we have proved that all eigenvalues of the problem (1.1)–(1.5) are real and the eigenfunctions corresponding to the different eigenvalues are orthogonal in the following sense:

$$\int_{-1}^{0} u_1(x)u_2(x)dx + h_1h_2\int_0^1 u_1(x)u_2(x)dx + \frac{h_1h_2}{ab}(u_1'(1) + bu_1(1))(u_2'(1) + bu_2(1)) = 0.$$

2. Construction and asymptotic approximation of the basic solutions

Since the function $q(x)$ is continuous on $[-1,0)$ and has a finite left-hand-side limit $q(-0)$, the Cauchy problem

$$\begin{gathered} -u'' + q(x)u = \lambda u, \quad x \in (-1,0), \\ u(-1) = 0, \quad u'(-1) = 1 \end{gathered} \tag{2.1}$$

has a unique solution $\phi_1(x,\lambda)$ which is an entire function of λ for each $x \in [-1,0)$. (See, for example, [10, Theorem 1.5].)

Now we can define the solution $\phi_2(x,\lambda)$ of equation (1.1) on $(0,1]$ by the initial conditions

$$u(+0) = \frac{1}{h_1}\phi_1(-0,\lambda), \quad u'(+0) = \frac{1}{h_2}\phi_1'(-0,\lambda). \tag{2.2}$$

In [12] we proved that this solution also is an entire function of λ for each $x \in (0,1]$.

Consequently, the function $\phi(x,\lambda)$ defined on $[-1,0)\cup(0,1]$ by

$$\phi(x,\lambda) = \begin{cases} \phi_1(x,\lambda), & \text{for } x \in [-1,0) \\ \phi_2(x,\lambda), & \text{for } x \in (0,1] \end{cases}$$

is such a solution of equation (1.1) on the whole of $[-1,0)\cup(0,1]$, which satisfies one of the boundary conditions (namely (1.2)) and both transmission conditions (1.4) and (1.5).

To construct the another basic solution

$$\chi(x,\lambda) = \begin{cases} \chi_1(x,\lambda), & \text{for } x \in [-1,0) \\ \chi_2(x,\lambda), & \text{for } x \in (0,1] \end{cases}$$

of the problem (1.1)–(1.5), we first define the solution $\chi_2(x,\lambda)$ of equation (1.1) on $(0,1]$ by initial conditions

$$u(1) = \lambda - a, \quad u'(1) = -b\lambda \tag{2.3}$$

and then the solution $\chi_1(x,\lambda)$ of equation (1.1) on $[-1,0)$ by initial conditions

$$u(-0) = h_1\chi_2(+0,\lambda), \quad u'(-0) = h_2\chi_2'(+0,\lambda). \tag{2.4}$$

Applying the same technique as in [1] we can prove that $\chi_1(x,\lambda)$ and $\chi_2(x,\lambda)$ are entire functions of λ for each fixed x in $[-1,0)$ and $(0,1]$, respectively.

Consequently, the function $\chi(x,\lambda)$ is a solution of equation (1.1) on $[-1,0)\cup(0,1]$, which satisfies the other boundary condition (1.3) and both transmission conditions (1.4) and (1.5).

LEMMA 1: *The solution $\phi(x,\lambda)$ satisfies the following integral equalities:*

$$\begin{aligned}\phi(x,\lambda) =&\phi(c_i,\lambda)\cos[s(x-c_i)]+\frac{1}{s}\phi'(c_i,\lambda)\sin[s(x-c_i)]\\&+\frac{1}{s}\int_{c_i}^{x}\sin[s(x-y)]q(y)\phi(y,\lambda)dy,\quad x\in(c_i,d_i),\quad i=1,2\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}\phi'(x,\lambda) =&-s\phi(c_i,\lambda)\sin[s(x-c_i)]+\phi'(c_i,\lambda)\cos[s(x-c_i)]\\&+\int_{c_i}^{x}\cos[s(x-y)]q(y)\phi(y,\lambda)dy,\quad x\in(c_i,d_i),\quad i=1,2,\end{aligned}\tag{2.6}$$

where $\lambda=s^2$, $c_1=-1$, $d_1=-0$, $c_2=+0$, $d_2=1$.

Proof: Substituting the identity $q(y)\phi(y,\lambda)=\phi''(y,\lambda)+s^2\phi(y,\lambda)$ in the right side of (2.5) we have

$$\begin{aligned}\frac{1}{s}\int_{c_i}^{x}\sin[s(x-y)]q(y)\phi(y,\lambda)dy =&\frac{1}{s}\int_{c_i}^{x}\sin[s(x-y)]\phi''(y,\lambda)dy\\&+s\int_{c_i}^{x}\sin[s(x-y)]\phi(y,\lambda)dy.\end{aligned}$$

On integrating by parts twice and using (2.1) and (2.2) we have

$$\begin{aligned}\int_{c_i}^{x}\sin[s(x-y)]\phi''(y,\lambda) =&s\phi(x,\lambda)-\phi'(c_i,\lambda)\sin[s(x-c_i)]\\&-s\phi(c_i,\lambda)\cos[s(x-c_i)]\\&-s^2\int_{c_i}^{x}\sin[s(x-y)]\phi(y,\lambda)dy.\end{aligned}$$

Substituting this back into the previous equality yields the needed (2.5).

The formula (2.6) is obtained by differentiating (2.5). ∎

In order to investigate the asymptotic bahaviour of the eigenvalues and corresponding eigenfunctions we shall need the following basic theorem.

THEOREM 1: *Let $\lambda=s^2$ and $s=\sigma+it$. The following asymptotic formulas are satisfied as $|\lambda|\to+\infty$:*

$$\phi_1(x,\lambda)=\frac{1}{s}\sin[s(x+1)]+O\Big(\frac{1}{|s|^2}e^{|t|(x+1)}\Big),\tag{2.7}$$

$$\phi_1'(x,\lambda)=\cos[s(x+1)]+O\Big(\frac{1}{|s|}e^{|t|(x+1)}\Big),\tag{2.8}$$

$$\phi_2(x,\lambda)=\frac{1}{h_1}\frac{1}{s}\sin(s)\cos(sx)+\frac{1}{h_2}\frac{1}{s}\cos(s)\sin(sx)+O\Big(\frac{1}{|s|^2}e^{|t|(x+1)}\Big)\tag{2.9}$$

and

$$\phi_2'(x,\lambda) = -\frac{1}{h_1}\sin(s)\sin(sx) + \frac{1}{h_2}\cos(s)\cos(sx) + O\Big(\frac{1}{|s|}e^{|t|(x+1)}\Big). \tag{2.10}$$

Moreover, each of these asymptotic equalities holds uniformly for x.

Proof: The proof of (2.7) and (2.8) is totally similar to the proof of Titchmarsh's lemma [10, Lemma 1.7]. But the proof of (2.9) and (2.10) needs individual consideration, since the solution $\phi_2(x,\lambda)$ is defined by means of $\phi_1(x,\lambda)$ by the special type initial conditions (2.2). By Lemma 1 we have

$$\begin{aligned}\phi_2(x,\lambda) = \phi_2(+0,\lambda)\cos(sx) + \frac{1}{s}\phi_2'(+0,\lambda)\sin(sx)\\ + \frac{1}{s}\int_0^x \sin[s(x-y)]q(y)\phi_2(y,\lambda)dy.\end{aligned} \tag{2.11}$$

Making use of the definition of $\phi_2(x,\lambda)$ and the asymptotic formulas (2.7) and (2.8) we have

$$\phi_2(+0,\lambda) = \frac{1}{h_1}\phi_1(-0,\lambda) = \frac{1}{h_1}\frac{1}{s}\sin(s) + O(|s|^{-2}e^{|t|}),$$

and similarly

$$\phi_2'(+0,\lambda) = \frac{1}{h_2}\cos(s) + O(|s|^{-1}e^{|t|}).$$

Substituting back into (2.11) we have the next "asymptotic integral equation" for $\phi_2(x,\lambda)$:

$$\begin{aligned}\phi_2(x,\lambda) =& \frac{1}{h_1}\frac{1}{s}\sin(s)\cos(sx) + \frac{1}{h_2}\frac{1}{s}\cos(s)\sin(sx)\\ &+ \frac{1}{s}\int_0^x \sin[s(x-y)]\phi_2(y,\lambda)dy + O(|s|^{-2}e^{|t|(x+1)}).\end{aligned} \tag{2.12}$$

Multiplying both sides by $se^{-|t|(x+1)}$ and letting

$$\varphi(x,\lambda) := se^{-|t|(x+1)}\phi_2(x,\lambda)$$

gives

$$\begin{aligned}\varphi(x,\lambda) =& \Big[\frac{1}{h_1}\sin(s)\cos(sx) + \frac{1}{h_2}\cos(s)\sin(sx)\Big]e^{-|t|(x+1)}\\ &+ \frac{1}{s}\int_0^x \sin[s(x-y)]e^{-|t|(x-y)}q(y)\varphi(y,\lambda)dy + O\Big(\frac{1}{|s|}\Big)\end{aligned}$$

as $|\lambda| \to \infty$.

Let

$$\varphi(\lambda) := \max_{x\in(0,1]} |\varphi(x,\lambda)|.$$

Then from the last asymptotic equality it follows that

$$\varphi(\lambda) \le \frac{1}{|h_1|} + \frac{1}{|h_2|} + \frac{Q}{|s|}\varphi(\lambda) + O\Big(\frac{1}{|s|}\Big)$$

as $|\lambda| \to \infty$, where $Q = \int_0^1 |q(y)|dy$. So, for $|s| > 2Q$ we have

$$\varphi(\lambda) < 2\Big(\frac{1}{|h_1|} + \frac{1}{|h_2|} + O\Big(\frac{1}{s}\Big)\Big).$$

Consequently, $\varphi(\lambda) = O(1)$ as $|\lambda| \to \infty$ and so

$$\phi_2(x,\lambda) = O(|s|^{-1}e^{|t|(x+1)}).$$

Substituting into the integral term of (2.12) gives

$$\int_0^x \sin[s(x-y)]q(y)\phi_2(y,\lambda)dy = O(|s|^{-1}e^{|t|(x+1)}).$$

Finally, by putting this formula in (2.12) we have the required asymptotic equality (2.9).

Similarly, we can prove the asymptotic equality (2.10). This concludes the proof of Theorem 1. ∎

3. Asymptotic formulae for eigenvalues

It is well-known from ordinary linear differential equations theory that each of the wronskians

$$\omega_1(\lambda) := W(\phi_1(x,\lambda),\chi_1(x,\lambda)) \quad \text{and} \quad \omega_2(\lambda) := W(\phi_2(x,\lambda),\chi_2(x,\lambda))$$

are independent of x in $[-1,0)$ and $(0,1]$, respectively. In [12] we proved that they are entire functions of λ. By using the initial conditions (2.2) and (2.4) we have

$$\begin{aligned} W(\phi_1(x,\lambda),\chi_1(x,\lambda)) &= W(\phi_1(0,\lambda),\chi_1(0,\lambda)) \\ &= h_1h_2W(\phi_2(0,\lambda),\chi_2(0,\lambda)), \end{aligned}$$

so

$$\omega_1(\lambda) = h_1h_2\omega_2(\lambda).$$

Set

$$\omega(\lambda) := \omega_1(\lambda) = h_1h_2\omega_2(\lambda).$$

In [12] we proved that the eigenvalues of the boundary-value transmission problem (1.1)–(1.5) are real and coincide with the zeros of the boundary function $\omega(\lambda)$. Consequently, there are finite or numerable many eigenvalues of the problem (1.1)–(1.5) since $\omega(\lambda)$ is an entire function.

We shall now concern ourselves with this boundary function $\omega(\lambda)$.

Lemma 2: *The boundary function $\omega(\lambda)$ has the following asymptotic representation:*

$$\omega(\lambda) = -\frac{h_1+h_2}{2}s^2\Big(\cos 2s - \frac{h_1-h_2}{h_1+h_2}\Big) + O(|s|e^{2|t|}), \ \text{as } |\lambda|\to\infty. \tag{3.1}$$

Proof: By using the initial values (2.3) of $\chi(x,\lambda)$ we have

$$\begin{aligned}\omega(\lambda) &= h_1h_2W_\lambda(\phi,\chi;1) = h_1h_2[\phi(1,\lambda)\chi'(1,\lambda)-\phi'(1,\lambda)\chi(1,\lambda)]\\ &= h_1h_2[-\lambda\phi(1,\lambda)-(\lambda-a)\phi'(1,\lambda)].\end{aligned} \tag{3.2}$$

On the other hand, by putting $x=1$ in (2.9) and (2.10) we have

$$\phi(1,\lambda) = \frac{1}{s}\frac{h_1+h_2}{2h_1h_2}\sin 2s + O(|s|^{-2}e^{2|t|}) \tag{3.3}$$

and

$$\phi'(1,\lambda) = \frac{h_1+h_2}{2h_1h_2}\Big[\cos 2s - \frac{h_1-h_2}{h_1+h_2}\Big] + O(|s|^{-1}e^{2|t|}), \tag{3.4}$$

respectively.

Finally, by substituting these formulas into (3.2) we get the required representation (3.1). ∎

Lemma 3: *The eigenvalues of the problem (1.1)–(1.5) are bounded below.*

Proof: If we suppose to the contrary, then there exists a sequence of eigenvalues $\{\lambda_n\}$ such that $\lambda_n\to-\infty$ as $n\to\infty$. Denoting

$$\lambda_n = -\mu_n^2$$

and substituting in (3.1) we get

$$\mu_n^2\Big(\cos 2i\mu_n - \frac{h_1-h_2}{h_1+h_2}\Big) = O(|\mu_n|e^{2\mu_n}),$$

and so

$$\mu_n = O(1).$$

Thus we get a contradiction, which completes the proof . ∎

THEOREM 2: *The problem (1.1)–(1.5) has precisely numerably many real eigenvalues $\lambda_0, \lambda_1, \lambda_2, \dots$ ($\lambda_n = s_n^2$) whose behaviour at infinity is specified by the following formulas:*

Case 1. *If $h_1 = h_2$, then*

$$s_n = \frac{\pi}{2}\Big(n - \frac{1}{2}\Big) + O\Big(\frac{1}{n}\Big). \tag{3.5}$$

Case 2. *If $h_1 \neq h_2$, then*

$$s_n = \begin{cases} \pi k - \frac{1}{2}\arccos\frac{h_1-h_2}{h_1+h_2} + O(\frac{1}{n}), & \text{for } n = 2k, \\ \pi k - (\pi - \frac{1}{2}\arccos\frac{h_1-h_2}{h_1+h_2}) + O(\frac{1}{n}), & \text{for } n = 2k-1. \end{cases} \tag{3.6}$$

Proof: We shall give the details for the more complicated Case 2 only. Let $\lambda = s^2$ and $s = \sigma + it$. Denote by $\tilde{\omega_1}(s)$ and $\tilde{\omega_2}(s)$ the leading and O-term on the right of (3.1), respectively. It is obvious that $\tilde{\omega_1}(s)$ has zeros at

$$\tilde{s}_0^{\pm} = 0, \quad \tilde{s}_n^{\pm} = \pm\frac{1}{2}\arccos\frac{h_1 - h_2}{h_1 + h_2} + \pi(n-1), \quad n = \pm 1, \pm 2, \dots,$$

where each zero is counted according to its multiplicity.

Below, for convenience, we shall use the notation

$$\theta = \frac{1}{2}\arccos\frac{h_1 - h_2}{h_1 + h_2}.$$

Let $\delta > 0$ be any number for which

$$0 < \delta < \min\Big\{\theta, \frac{\pi}{2} - \theta\Big\} \tag{3.7}$$

and let $\Gamma_n^{\pm}$ be the bound of the domain

$$\{s = \sigma + it : |\sigma|, |t| \le \pm\theta + \pi(n-1) + \delta\}.$$

It is obvious from the definition of $\Gamma_n^{\pm}$ that $|\tilde{\omega_1}(s)| > |\tilde{\omega_2}(s)|$ on $\Gamma_n^{\pm}$ for sufficiently large n. Therefore, by the well-known Rouche's Theorem (which asserts that if $f(z)$ and $g(z)$ are analytic inside and on a closed contour Γ, and $|g(z)| < |f(z)|$ on Γ, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside Γ, provided that each zero is counted according to its multiplicity), the function $\omega(s^2) = \tilde{\omega_1}(s) + \tilde{\omega_2}(s)$ has precisely $4n-2$ zeros inside Γ^- and $4n$ zeros inside Γ^+, so that between $\Gamma_n^{\pm}$ and $\Gamma_{n+1}^{\pm}$ there is precisely one positive zero for sufficiently large n. Since the function $\omega(s^2)$ is even, it is enough only to derive an asymptotic formula for positive eigenvalues. We observe that the zeros $\{s_n\}$ of $\omega(s^2)$ may be represented as two subsequences s_{2n-1} and s_{2n-2} of the forms

$$s_{2n-2} = \tilde{s}_n^- + \delta_n^- = -\theta + \pi(n-1) + \delta_n^- \tag{3.8}$$

and

$$s_{2n-1} = \tilde{s}_n^+ + \delta_n^+ = \theta + \pi(n-1) + \delta_n^+,$$

where $|\delta_n^\pm| < \delta$.

Let us show that

$$\delta_n^\pm = O\Big(\frac{1}{n}\Big).$$

Putting (3.8) in (3.1) we have

$$s_{2n-2}^2\Big[\cos(-2\theta + 2\pi(n-1) + 2\delta_n^-) - \frac{h_1 - h_2}{h_1 + h_2}\Big] = O(|s_{2n-2}|).$$

From this, by (3.7) it follows that

$$\sin(-2\theta + \delta_n^-)\sin\delta_n^- = O\Big(\frac{1}{n}\Big). \tag{3.9}$$

Further, in view of (3.7) we see that there is $A > 0$ such that

$$\sin(-2\theta + \delta_n^+) > A$$

for sufficiently large n, and so

$$\sin\delta_n^- = O\Big(\frac{1}{n}\Big).$$

Again by (3.7), this asymptotic equality implies that

$$\delta_n^- = O\Big(\frac{1}{n}\Big). \tag{3.10}$$

Similarly,

$$\delta_n^+ = O\Big(\frac{1}{n}\Big). \tag{3.11}$$

Thus the proof for Case 2 is complete.

The argument for Case 1 is analogous. ∎

4. Asymptotic formulae for eigenfunctions

Let $\phi(x,\lambda)$ be defined as in Section 2. We already know that $\phi(x,\lambda_n)$ is an eigenfunction corresponding to the eigenvalue λ_n. Let $\phi_n(x) = \phi(x,\lambda_n)$.

THEOREM 3: *The eigenfunctions $\phi_n(x)$ have the following asymptotic representation as $n \to \infty$:*

Case 1*: If $h_1 = h_2$, then*

(4.1)
$$\phi_n(x) = \begin{cases} [\frac{\pi}{2}(n-\frac{1}{2})]^{-1}\sin[\frac{\pi}{2}(n-\frac{1}{2})(x+1)] + O(\frac{1}{n}), & \text{for } x \in [-1,0), \\ \frac{1}{h}[\frac{\pi}{2}(n-\frac{1}{2})]^{-1}\sin[\frac{\pi}{2}(n-\frac{1}{2})(x+1)] + O(\frac{1}{n}), & \text{for } x \in (0,1], \end{cases}$$

where $h := h_1 = h_2$.

Case 2*: If $h_1 \neq h_2$, then for even n, $n = 2k$,*

(4.2)
$$\phi_n(x) = \begin{cases} (\pi k-\theta)^{-1}\sin[(\pi k-\theta)(x+1)] + O(\frac{1}{n^2}), \quad \text{for } x \in [-1,0), \\ \frac{(-1)^{k+1}(\pi k-\theta)^{-1}}{\sqrt{h_1+h_2}}\{\frac{\sqrt{h_2}}{h_1}\cos[(\pi k-\theta)x] - \frac{\sqrt{h_1}}{h_2}\sin[(\pi k-\theta)x]\} + O(\frac{1}{n^2}), \\ \qquad\qquad \text{for } x \in (0,1]. \end{cases}$$

and for odd n, $n = 2k-1$,

(4.3)
$$\phi_n(x) = \begin{cases} [\pi k-(\pi-\theta)]^{-1}\sin[(\pi k-(\pi-\theta))(x+1)] + O(\frac{1}{n^2}), \ \text{for } x \in [-1,0), \\ \frac{(-1)^{k+1}(\pi k-(\pi-\theta))^{-1}}{\sqrt{h_1+h_2}}\{\frac{\sqrt{h_2}}{h_1}\cos[(\pi k-(\pi-\theta))x] \\ \qquad + \frac{\sqrt{h_1}}{h_2}\sin[(\pi k-(\pi-\theta))x]\} \\ \qquad + O(\frac{1}{n^2}), \qquad\qquad \text{for } x \in (0,1]. \end{cases}$$

Proof: We shall give the details for Case 2, which is the more complicated of the two. Putting $\lambda = \lambda_n$ in (2.7) and (2.9) gives

$$\phi_1(x,\lambda_n) = \frac{1}{s_n}\sin[s_n(x+1)] + O(|s_n|^{-2}) \tag{4.4}$$

and

$$\phi_2(x,\lambda_n) = \frac{1}{s_n}\Big[\frac{1}{h_1}\sin(s_n)\cos(s_nx) + \frac{1}{h_2}\cos(s_n)\sin(s_nx)\Big] + O(|s_n|^{-2}), \tag{4.5}$$

respectively.

Let us consider the case $n = 2k$. In this case, from (3.6) we obtain the equalities

$$\begin{aligned} \frac{1}{s_{2k}} &= \Big(\pi k-\theta+O\Big(\frac{1}{k}\Big)\Big)^{-1} \\ &= \frac{1}{\pi k-\theta} + O\Big(\frac{1}{k^3}\Big), \end{aligned} \tag{4.6}$$

$$\begin{aligned} \sin(s_{2k}x) &= \sin\Big(\pi k-\theta+O\Big(\frac{1}{k}\Big)\Big)x \\ &= \sin(\pi k-\theta)x + O\Big(\frac{1}{k}\Big), \end{aligned} \tag{4.7}$$

and

$$\cos(s_{2k}x) = \cos(\pi k - \theta)x + O\Big(\frac{1}{k}\Big). \tag{4.8}$$

Putting $x = 1$ in (4.7) and (4.8) yields

$$\begin{aligned} \sin s_{2k} &= \sin(\pi k - \theta) + O\Big(\frac{1}{k}\Big) \\ &= (-1)^{k+1}\sin(\theta) + O\Big(\frac{1}{k}\Big) \\ &= (-1)^{k+1}\sqrt{\frac{h_2}{h_1+h_2}} + O\Big(\frac{1}{k}\Big) \end{aligned} \tag{4.9}$$

and

$$\cos s_{2k} = (-1)^k\sqrt{\frac{h_1}{h_1+h_2}} + O\Big(\frac{1}{k}\Big), \tag{4.10}$$

respectively.

Substituting these asymptotic equalities in (4.5) yields

$$\begin{aligned} \phi_2(x,\lambda_{2k}) =& \frac{(-1)^{k+1}}{\sqrt{h_1+h_2}}\frac{1}{\pi k - \theta}\Big\{\frac{\sqrt{h_2}}{h_1}\cos[(\pi k-\theta)x] - \frac{\sqrt{h_1}}{h_2}\sin[(\pi k - \theta)x]\Big\} \\ &+ O\Big(\frac{1}{k^2}\Big). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \phi_2(x,\lambda_{2k-1}) =& \frac{(-1)^{k+1}}{\sqrt{h_1+h_2}}\frac{1}{\pi k - (\pi-\theta)}\Big\{\frac{\sqrt{h_2}}{h_1}\cos[(\pi k-(\pi-\theta))x] \\ &+ \frac{\sqrt{h_1}}{h_2}\sin[(\pi k - (\pi-\theta))x]\Big\} \\ &+ O\Big(\frac{1}{k^2}\Big). \end{aligned}$$

Thus the proof for the case $h_1 \neq h_2$ is complete. The case $h_1 = h_2$ is proved similarly. ∎

5. Counterexample

All the results in this study are derived under the conditions $ab > 0$ and $h_1h_2 > 0$ on the coefficients of the boundary and transmission conditions. We can show that these conditions are essential, i.e., they cannot be omitted. In fact, let us consider the equation

$$-u'' = \lambda u, \quad x \in [-1,0) \cup (0,1] \tag{5.1}$$

with boundary conditions

$$u(-1) = 0, \quad (\lambda - 1)u'(1) + \lambda u(1) = 0 \tag{5.2}$$

and with transmission conditions

$$u(-0) = u(+0), \quad u'(-0) = -u'(+0), \tag{5.3}$$

for which the condition $h_1 h_2 > 0$ is not satisfied. Let $\lambda = s^2$ and $\lambda \neq 0$ (it is easy to show that $\lambda = 0$ is not an eigenvalue of this problem). The general solution of equation (5.1) on $[-1, 0) \cup (0, 1]$ is of the form

$$u(x, \lambda) = \begin{cases} C_1 \cos sx + C_2 \sin sx, & \text{for } x \in [-1, 0), \\ C_3 \cos sx + C_4 \sin sx, & \text{for } x \in (0, 1]. \end{cases} \tag{5.4}$$

Applying the boundary and transmission conditions (5.2) and (5.3) to the general solution (5.4), we obtain the homogeneous linear system of equations with respect to the variables C_1, C_2, C_3 and C_4. It is easy to show that the determinant of this system is equal to $s(s^2 - 1)$. Therefore, the problem (5.1)–(5.3) has only one eigenvalue $\lambda = 1$, for which the corresponding eigenfunction is of the form

$$u(x) = C \begin{cases} \sin(1 + x), & \text{for } x \in [-1, 0), \\ \sin(1 - x), & \text{for } x \in (0, 1], \end{cases}$$

where C is an arbitrary constant.

Consequently, our results are not valid in general when the condition $h_1 h_2 > 0$ is not satisfied.

6. Remark

It is well-known that the “standard” Sturm–Liouville problems have precisely numerable many eigenvalues and the leading terms of asymptotics of the eigenvalues and corresponding eigenfunctions are not dependent on the boundary conditions. In contrast to this fact, the leading terms of the asymptotics of the eigenvalues and corresponding eigenfunctions of our “non-standard” problem (1.1)–(1.5) are dependent on boundary conditions (1.4)–(1.5).

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